Quasi-Dirac operators and quasi-fermions

2009 J. Phys. A: Math. Theor. 42 155201

(http://iopscience.iop.org/1751-8121/42/15/155201)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 155201 (10pp) doi:10.1088/1751-8113/42/15/155201

Quasi-Dirac operators and quasi-fermions

Andrzej Sitarz[1]

Institute of Physics, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

Received 28 November 2008, in final form 2 February 2009
Published 19 March 2009
Online at stacks.iop.org/JPhysA/42/155201

Abstract
We investigate examples of quasi-spectral triples over a two-dimensional commutative sphere, which are obtained by modifying the order-one condition. We find equivariant quasi-Dirac operators and prove that they are in a topologically distinct sector from the standard Dirac operator.

PACS numbers: 02.40.Gh, 02.40.−k, 04.20.Gz
Mathematics Subject Classification: 58B34, 46L87, 34L40

1. Introduction

Noncommutative geometry offers a new insight into classical differential geometry of spin manifolds. The Connes' reconstruction theorem, which under certain assumptions is proved in [5] and has been recently extended [4], provides the equivalence between the spectral triples for commutative geometries and the geometry of spin manifolds.

In this paper, we investigate what happens if one relaxes one of the axioms of the spectral triple construction by allowing the order-one condition to be satisfied up to compact operators. This step is motivated by a strictly noncommutative q-deformed situation. In the case of $SU_q(2)$ [6, 8] and the Podleś equatorial quantum sphere [7], it was found that the order-one condition and the commutant condition can be satisfied only up to compact operators. It has been conjectured that for commutative geometries similar weakening of this axiom shall not lead to any significant change apart from small (compact) perturbation of the Dirac operator. In our example, we show that this is not true, in particular, the spectral geometries obtained in this way are topologically distinct.

For the two-dimensional sphere, we determine a class of examples of equivariant quasi-spectral geometries. Using local index calculations, we prove that they pair differently with an element of K-theory and therefore are mutually inequivalent.

The two-dimensional commutative sphere has been a favorite toy model for the description of noncommutative geometry but in a local (coordinate) description [5, 9]. A *global* approach was first used by Paschke [10] and could also be easily obtained by taking the $q = 1$ limit of the algebraic description of Podleś spheres. Our work is partially motivated by an attempt

[1] Partially supported by MNII grant 189/6.PRUE/2007/7.

to make sense of the classical limits of spectral geometries obtained for q-deformed spheres in [12].

2. Spectral geometry of the sphere

We follow here the construction of equivariant spectral geometry as described in [11]. The symmetry which we use is the enveloping algebra of $\mathfrak{su}(2)$, with generators L^3, $L^\pm$ satisfying

$$[L^\pm, L^3] = \mp L^\pm, \qquad [L^+, L^-] = 2L^3.$$

We work with the commutative star algebra $\mathcal{A}(S^2)$ of polynomials in $A, B, B^*, A = A^*$, satisfying the radius relation

$$A^2 + BB^* = 1. \tag{2.1}$$

The generators of the polynomial algebra A, B, B^* are the spherical harmonics of degree 1. The action of the $\mathfrak{su}(2)$ Lie algebra on the generators is

$$\begin{array}{lll} L^+ \triangleright B = 0, & L^- \triangleright B = -2A, & L^3 \triangleright B = B, \\ L^+ \triangleright B^* = 2A, & L^- \triangleright B^* = 0, & L^3 \triangleright B^* = -B^*, \\ L^+ \triangleright A = -B, & L^- \triangleright A = B^*, & L^3 \triangleright A = 0. \end{array}$$

2.1. The equivariant representations

Let V_l, $l = 0, \frac{1}{2}, 1, \ldots,$ denote the $(2l+1)$-dimensional representation of the Lie algebra $\mathfrak{su}(2)$. The orthonormal basis of each V_l shall be denoted by $|l, m\rangle$, $m = -l, -l+1, \ldots, l-1, l$. The representation ρ of $\mathfrak{su}(2)$ on V_l is

$$\begin{aligned} L^+|l, m\rangle &= \sqrt{l-m}\sqrt{l+m+1}|l, m+1\rangle, \\ L^-|l, m\rangle &= \sqrt{l+m}\sqrt{l-m+1}|l, m-1\rangle, \\ L^3|l, m\rangle &= m|l, m\rangle. \end{aligned}$$

We recall that for any algebra $\mathcal{A}$ on which a Lie algebra $\mathfrak{l}$ acts by derivation, the representation π is $\mathfrak{l}$-equivariant if there exists a representation ρ of $\mathfrak{l}$, such that

$$\rho(\ell)\pi(a) = \pi(\ell \triangleright a) + \pi(a)\rho(\ell) \tag{2.2}$$

holds for all $a \in \mathcal{A}$, $\ell \in \mathfrak{l}$ on the linear space $\mathcal{V}$.

Since we know the representation theory of $\mathfrak{su}(2)$, we shall decompose $\mathcal{V}$ as a direct sum of irreducible representations of $\mathfrak{su}(2)$:

$$\mathcal{V} = \bigoplus_l V_l.$$

We have

Proposition 2.1. *For each $N = 0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \ldots,$ there exists an irreducible unitary representation π_N of $\mathcal{A}(S^2)$ on the Hilbert space $\mathcal{H}_N$, which is the completion of $\mathcal{V}_N$:*

$$\mathcal{V}_N = \bigoplus_{l=|N|}^{\infty} V_l,$$

which is equivariant on $\mathcal{V}_N$ with respect to $\mathfrak{su}(2)$ action. The explicit form for π_N on the orthonormal basis, $|l, m\rangle \in V_l$, is

$$\begin{aligned}\pi_N(B)|l,m\rangle &= \sqrt{(l+m+1)(l+m+2)}\alpha_l^+|l+1,m+1\rangle \\ &\quad + \sqrt{(l+m+1)(l-m)}\alpha_l^0|l,m+1\rangle \\ &\quad - \sqrt{(l-m)(l-m-1)}\alpha_{l-1}^+|l-1,m+1\rangle,\end{aligned} \tag{2.3}$$

$$\begin{aligned}\pi_N(B^*)|l,m\rangle &= -\sqrt{(l-m+2)(l-m+1)}\alpha_l^+|l+1,m-1\rangle \\ &\quad + \sqrt{(l+m)(l-m+1)}\alpha_l^0|l,m-1\rangle \\ &\quad + \sqrt{(l+m)(l+m-1)}\alpha_{l-1}^+|l-1,m-1\rangle,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\pi_N(A)|l,m\rangle &= -\sqrt{(l-m+1)(l+m+1)}\alpha_l^+|l+1,m\rangle \\ &\quad + m\alpha_l^0|l,m\rangle \\ &\quad - \sqrt{(l-m)(l+m)}\alpha_{l-1}^+|l-1,m\rangle,\end{aligned} \tag{2.5}$$

where α_l^+, α_l^0 *are*

$$\alpha_l^0 = \frac{N}{l(l+1)}, \qquad \alpha_l^+ = \sqrt{1-\frac{N^2}{(l+1)^2}}\frac{1}{\sqrt{(2l+1)(2l+3)}}. \tag{2.6}$$

Proof. The proof is mostly technical, and we only sketch here its main points, especially since parts of it are equivalent to the Wigner–Eckart theorem on $su(2)$ irreducible tensors[2]. For simplicity, we write π instead of π_N, keeping in mind that the representation depends on N.

First, using the equivariance (2.2) for L^3 we see that $\pi(A)$ does not change the eigenvalue m, whereas $\pi(B), \pi(B^*)$ change it by ± 1. Then, we deduce that for $X = A, B, B^*$ and for each l we have

$$\pi(X)(V_l) \subset V_{l-1} \oplus V_l \oplus V_{l+1}, \tag{2.7}$$

where (*a priori*) we can have multiplicities on the right-hand side. To see that (2.7) holds, it suffices to study the equivariance rule for $\rho(L^{\pm})^n$ for a suitably chosen n. For instance, if $X = A$, then using the equivariance we verify that

$$\rho((L^+)^{l-m+1}\pi(A)|l,m\rangle = 0, \qquad \rho((L^-)^{l+m+1}\pi(A)|l,m\rangle = 0.$$

Therefore, $\pi(A)|l,m\rangle$ must be in the intersection of kernels of $\rho(L^+)^{l-m+1}$ and $\rho(L^-)^{l+m+1}$ with the eigenspace of $\rho(L^3)$ to the eigenvalue m. Therefore, the result (2.7) for $X = A$ becomes clear.

The equivariance rule for $L^{\pm}$ allows us to fix the m-dependence of the coefficients in the expansion (2.7). We again use the example of A to illustrate it:

$$\langle l+1, m+2|\rho(L^+)\pi(B)|l,m\rangle = \sqrt{l-m}\sqrt{l+m+3}\langle l+1,m+1|\pi(B)l,m\rangle,$$

where we used that $(L^+)^* = L^-$. On the other hand, using equivariance

$$\begin{aligned}\langle l+1, m+2|\rho(L^+)\pi(B)|l,m\rangle &= \langle l+1,m+2|\pi(B)\rho(L^+)|l,m\rangle \\ &= \sqrt{l-m}\sqrt{l+m+1}\langle l+1,m+2|\pi(B)l,m+1\rangle.\end{aligned}$$

By comparing these expressions, we see that $B_{l,m}^+ = \langle l+1,m+1|\pi(B)|l,m\rangle$ must satisfy

$$B_{l,m}^+\sqrt{l+m+3} = B_{l,m+1}^+\sqrt{l+m+1}, \tag{2.8}$$

[2] I would like to thank F J Vanhecke for raising this point.

and it has the solution

$$B^{+}_{l,m} = \sqrt{l+m+1}\sqrt{l+m+2}\alpha^{+}_{l}.$$

Similarly we find the other remaining coefficients.

The recurrence relations for α^{+}_{l} and α^{0}_{l} are obtained from the sphere radius restriction (2.1). The freedom in the choice of the free parameters in their solutions corresponds to the choice of the smallest possible $|N|$, whereas the sign of α^{0}_{l} fixes the sign of N. □

2.2. *Equivariant spectral geometry*

To construct the spectral geometry following the axioms [3], in addition to the equivariant representation, we need a $\mathbb{Z}_2$-grading γ of the Hilbert space and the reality operator J, satisfying $JD = DJ$ and

$$J\gamma = -\gamma J, \qquad J^2 = -1, \qquad \gamma^2 = 1, \tag{2.9}$$

with the equivariance conditions

$$\rho(\ell)\gamma = \gamma\rho(\ell), \qquad \forall\, \ell \in \mathfrak{su}(2), \tag{2.10}$$

$$J\rho(\ell) = -\rho(\ell)^* J, \qquad \forall\, \ell \in \mathfrak{su}(2). \tag{2.11}$$

It follows directly from (2.10) that γ must be diagonal with respect to the chosen basis on $\mathcal{H}_N$. Additionally, if it commutes with the representation π_N of $\mathcal{A}(S^2)$ then it must be either 1 or -1. The Hilbert space with which we construct the spectral geometry must be a direct sum of at least two spaces $\mathcal{H}_N$ and $\mathcal{H}'_N$ with γ being ± 1 on them, respectively.

Further, the existence of an antilinear equivariant isometry, J, fixes $|N| = |N'|$. This is so, because the commutation rule (2.9) requires that J is the isometry between $\mathcal{H}_N$ and $\mathcal{H}'_N$. Then the equivariance condition (2.11) requires that J must map $|l,m\rangle$ to $|l,-m\rangle$, for any l, but this is possible only if $|N| = |N'|$.

It remains to fix the representation (or, in other words, the sign of N, N'). This follows from a further requirement, the commutant condition

$$J\pi(x)J^{-1} = \pi(x^*), \qquad \forall\, x \in \mathcal{A}(S^2), \tag{2.12}$$

which leads to the solution $\mathcal{H} = \mathcal{H}_N \oplus \mathcal{H}_{-N}, N \geqslant 0$. We shall denote the basis by $|l,m,\pm\rangle \in \mathcal{H}_{\pm N}$. So our spectral data so far consist of $\mathcal{H}$, the diagonal representation of $\mathcal{A}(S^2)$ on it and the operators J, γ:

$$\gamma|l,m,\pm\rangle = \pm|l,m,\pm\rangle, \tag{2.13}$$

$$J|l,m,\pm\rangle = i^{2m}|l,-m,\mp\rangle. \tag{2.14}$$

Observe that only for a half-integer N we have the signs of a two-dimensional spectral geometry, whereas for an integer value of N, we have the sign relations corresponding formally to a six-dimensional (mod 8) real structure.

2.3. *The equivariant Dirac operator*

We assume that D is a densely defined symmetric operator, equivariant with respect to $su(2)$ symmetry and satisfying the commutation requirements of spectral geometry in dimension 2:

$$DJ = JD, \qquad D\gamma = -\gamma D.$$

It is easy to see that any operator with these properties must be of the form

$$D|l, m, \pm\rangle = d(l)^{\pm}|l, m, \mp\rangle, \tag{2.15}$$

where $(d(l)^{\pm})^* = d(l)^{\mp}$ are complex coefficients.

The most important restriction comes from the order-one condition, which in its exact form reads (for commutative geometries)

$$[[D, \pi(x)], \pi(y)] = 0, \qquad \forall\, x, y \in \mathcal{A}(S^2). \tag{2.16}$$

Proposition 2.2. *The order-one condition is satisfied exactly only if $N = \frac{1}{2}$ and $d(l) = (l+\frac{1}{2})d_1$ for any $d_1 \neq 0$. For the same form of the Dirac operator but different N the order-one condition is satisfied up to compact operators $\mathcal{K}(\mathcal{H})$:*

$$[[D, \pi(x)], \pi(y)] \in \mathcal{K}(\mathcal{H}), \qquad \forall\, x, y \in \mathcal{A}(S^2). \tag{2.17}$$

Proof. We start to check the order-one condition on the generators A, B, B^*. First, looking at the coefficient at $|l, m, +\rangle$ of $[\pi(B^*), [D, \pi(B)]]|l, m, +\rangle$, we obtain that it contains

$$\frac{-4}{(2l-1)(2l+1)(l+1)^2(2l+3)l^2}(2d^+(l) - d^+(l+1) - d(l-1))l^7 + \cdots,$$

where the remaining terms are of lower order in l.

Therefore, if we require that the commutator is compact, then $2d^+(l)-d^+(l+1)-d^+(l-1)$ must converge to 0 if $l \to \infty$. On the other hand, we are looking for an unbounded operator D, so $d^+(l)$ itself must diverge. We can solve the recurrence relation obtained by setting that $2d^+(l) - d^+(l+1) - d^+(l-1)$ vanishes, this gives

$$d^+(l) = d_1(l + d_0),$$

for an arbitrary real d_0. Of course, we can perturb the above solution by adding to $d^+(l)$ anything that converges to 0. Moreover, κ corresponds to a perturbation of the Dirac operator by its sign. For this reason, it is convenient to fix it so that in the *classical* case $\left(N = \frac{1}{2}\right)$ the order-one condition is satisfied exactly. A simple calculation shows that $d_0 = \frac{1}{2}$ is the desired value[3].

Then, choosing

$$d^+(l) = d_1 \cdot \left(l + \tfrac{1}{2}\right), \tag{2.18}$$

we know that the order-one condition is satisfied exactly in the $N = \frac{1}{2}$ case and at least some of the coefficients of $[\pi(B^*), [D, \pi(B)]]|l, m, +\rangle$ tend to 0 as $l \to \infty$.

However, putting (2.18) back into the expression $[\pi(B^*), [D, \pi(B)]]|l, m, +\rangle$, we obtain that it gives

$$[\pi(B^*), [D, \pi(B)]]|l, m, +\rangle = 4d_1 \frac{(4N^2-1)(2l^2+2l-1-2m^2)}{(2l-1)(2l+1)(2l+3)}|l, m, -\rangle.$$

Although this does not vanish for $N \neq \frac{1}{2}$ we can easily see that then $[\pi(B^*), [D, \pi(B)]]$ is a compact operator.

For large l we have

$$\|[\pi(B^*), [D, \pi(B)]]|l, m, +\rangle\| \leqslant 4|d_1||4N^2 - 1|\frac{1}{l}.$$

[3] Most of the calculations were checked using Maple. The source program is available on the author's homepage: http://th-www.if.uj.edu.pl/~sitarz/maple.html.

If we denote by Υ the isometry $\Upsilon|l,m,\pm\rangle = |l,m,\mp\rangle$, the operator $\Upsilon[\pi(B^*), [D,\pi(B)]]$ is diagonal and its sequence of eigenvalues converges to 0. Therefore it is compact and hence $[\pi(B^*),[D,\pi(B)]]$ is compact.

Similarly, we check other commutators, for instance,

$$[\pi(A),[D,\pi(B)]]|l,m,+\rangle = -4\frac{\sqrt{l-m}\sqrt{l+m+1}(4N^2-1)(2m+1)}{(4l^2-1)(2l+3)}|l,m,-\rangle,$$

and we find that one can always majorize them by $C\frac{1}{l}$, hence (2.17) holds. □

Note that the modified order-one condition does not really enforce the value of the coefficient $d_0 = \frac{1}{2}$ in formula (2.18) and could be replaced by an arbitrary coefficient. This freedom in the choice of this component d_0 does not play a role as it is a perturbation of the Dirac operator by its sign.

2.4. *The properties of the quasi-Dirac operators*

We shall briefly discuss the properties of the quasi-Dirac operators we have found, for an arbitrary N. For simplicity we fix d_1 to be 1. First, observe that apart from the first $2(N-1)$ eigenvalues, it has the same spectrum as the standard Dirac operator and therefore the asymptotic behavior of the eigenvalues is exactly the same. In particular, $|D|^{-2}$ is Dixmier class and its Dixmier trace is

$$\mathrm{Tr}_\omega|D|^{-2} = 2. \tag{2.19}$$

Clearly, all commutators $[D,x]$ for $x\in\mathcal{A}(S^2)$ are bounded operators (it is an easy exercise to verify it); therefore, from the point of view of the local index calculations, for each N we indeed have a two-dimensional spectral geometry.

2.5. *Local index calculation*

To get insight into the presented construction we shall explicitly calculate here the index pairing between the K-homology element represented by the construction of the spectral geometry and a chosen element of K-theory using the Connes–Moscovici [2] local index formula.

Let us recall that having the spectral data we can explicitly assign to the spectral geometry a cyclic cocycle from the b–B bicomplex. For the interesting case of a two-cyclic cocycle for the two-dimensional spectral geometry we have the following even b–B cocycle:

$$\phi_0(a_0) = \tau_{-1}(\gamma\pi(a_0)), \tag{2.20}$$

$$\phi_2(a_0,a_1,a_2) = \tfrac{1}{2}\tau_0(\gamma\pi(a_0)[D,\pi(a_1)][D,\pi(a_2)]|D|^{-2}), \tag{2.21}$$

where τ_q is defined as

$$\tau_q(T) = \mathrm{Res}_{z=0}\,\mathrm{Tr}(z^qT|D|^{-2z}).$$

The pairing of the above cocycle with the projector e depends only on the class of ϕ and e:

$$\langle[\phi],[e]\rangle = (2\pi\mathrm{i})^2\big(\phi_0(e) - 2\phi_2\big(e-\tfrac{1}{2},e,e\big)\big),$$

where the coefficient is chosen so that the pairing is integral. Choosing e:

$$e = \frac{1}{2}\begin{pmatrix}1-A & B\\ B^* & 1+A\end{pmatrix},$$

which is a representative of a nontrivial K-theory class of the sphere we have

$$\begin{aligned}\phi_0(e) &= \phi_0(\pi(e_{11})) + \phi_0(\pi(e_{22})) = \phi_0(1)\\ &= \tau_{-1}(\gamma) = 0,\end{aligned}$$

and

$$\phi_2\left(e - \frac{1}{2}, e, e\right) = \sum_{i,j,k=1,2} \phi_2\left(e_{ij} - \frac{1}{2}\delta_{ij}, e_{jk}, e_{ki}\right).$$

The calculation itself is a technical and rather tedious task. The crucial point is the following result:

$$\left(\sum_{i,j,k=1,2} \left(\pi(e_{ij}) - \frac{1}{2}\delta_{ij}\right)[D, \pi(e_{jk})][D, \pi(e_{ki})]\right) |l, m, \pm\rangle = \pm N |l, m, \pm\rangle, \tag{2.22}$$

from which we easily get

Proposition 2.3. *The pairing between the Chern character of the spectral geometry* $(\mathcal{A}(S^2), \mathcal{H}_N \oplus \mathcal{H}_{-N}, D)$ *and the element e of the K-theory is* $2N$.

Proof. Using (2.22) we explicitly calculate the pairing:

$$\begin{aligned}(2\pi \mathrm{i})^2\langle(\phi_0, \phi_2), e\rangle &= (2\pi \mathrm{i})^2\left(\phi_0(e) - 2\phi_2\left(e - \frac{1}{2}, e, e\right)\right)\\ &= -(2\pi \mathrm{i})^2 \frac{1}{2} 2N \mathrm{Res}_{z=0} \mathrm{Tr}(|D|^{-2-2z})\\ &= -2N(2\pi \mathrm{i})^2 \frac{1}{2(2\pi)^2} \mathrm{Tr}_\omega(|D|^{-2})\\ &= 4N(2\pi)^2 \frac{1}{2(2\pi)^2} = 2N,\end{aligned}$$

where we used further the relation between the residue and the Dixmier trace as well as the result (2.19). □

Observe that as a byproduct of the calculations we have verified the Hochschild cycle axiom of the spectral triple, as one can easily see from (2.22).

It appears that the calculation of the explicit formulae for the nonlocal Connes–Chern character using the Chern character in periodic cyclic cohomology, for a two-cyclic cocycle, is also possible, although it is (obviously) more complicated. Taking $F = \mathrm{sign}(D)$ we have

$$\Phi_2(a_0, a_1, a_2) = \tfrac{1}{4}\mathrm{Tr}(\gamma F[F, \pi(a_0)][F, \pi(a_1)][F, \pi(a_2)]).$$

After long calculations we obtain

$$\begin{aligned}\mathrm{Tr}\left(\sum_{i,j,k=1,2} \gamma F[F, e_{ij}][F, e_{jk}][F, e_{ki}]\right) &= 4N^3 \sum_{l=N,N+1,\ldots} \sum_{m=-l}^{l} \frac{1}{l^2(l+1)^2}\\ &= 4N^3\left(\sum_{l=N}^{\infty} \frac{2l+1}{l^2(l+1)^2}\right) = 4N,\end{aligned}$$

so the pairing, which could be calculated as $(-2)\Phi_2(e, e, e)$, becomes

$$\langle\Phi_2, e\rangle = -2\Phi_2(e, e, e) = 2N.$$

3. Quasi-fermions and differential operators

In the previous section, we have demonstrated that the Dirac operator D belongs to a different topological class. Now, we can ask, what would be the physical properties of fermions whose dynamics would be governed by such a *quasi-Dirac*.

Consider, for example, extension of the discussed Dirac operator D to three dimensions and the Dirac equation with a central potential. It is easy to see that the only significant difference would be the change in the dependence of the energy levels on the angular momentum compared to the standard case. The lowest angular momenta up to $N-1$ shall be absent from the energy spectrum. A naive physical interpretation of that might be that we have a description of fermions with spin different than $\frac{1}{2}$, thus raising questions of relations with the Rarita–Schwinger equation for spin $\frac{3}{2}$ particles. The difference between the two is, however, fundamental as our quasi-Dirac is necessarily a pseudodifferential operator, and therefore the Dirac equation is not a first-order differential equation.

The quasi-fermions might be, similarly to the usual ones, charged, thus interacting through a $U(1)$ gauge potential. Interestingly, since the gauge potential corresponds to a one-form and these shall not commute with the elements of the algebra, we shall obtain a mildly noncommutative gauge field theory. Whether this self-interaction of gauge potential shall appear in the spectral action is yet to be determined. Nevertheless, we expect that this mild type of noncommutativity can bring some interesting new effects, which we shall study in our future work.

The classical equivariant Dirac operator on the Riemannian sphere, which corresponds to the $N=\frac{1}{2}$ situation described above, gives a representative of a K-homology class. The standard method to obtain representatives of the remaining classes is to twist the Dirac spinor bundle S, by tensoring it (over the algebra of functions) with a nontrivial line bundle E over the sphere. Then, the twisted Dirac operator, D_E, is associated with the tensor product connection:

$$\nabla = \nabla_S \otimes 1 + 1 \otimes \nabla_E,$$

where ∇_S is the spin connection, and ∇_E is a connection on the line bundle E (for the details of construction, explicit formulae and properties, see [9, 13] and references therein).

The twisted Dirac operator D_E is again an elliptic, self-adjoint $\mathbb{Z}_2$-graded differential operator, and from the Atiyah–Singer theorem [1] we know that the index of D_E^+ depends on the Chern number k of the line bundle $E(k)$. Thus, twisting by inequivalent line bundles we obtain inequivalent Fredholm modules and thus different representatives of the K-homology group of the sphere. However, we can easily see that the construction does not resemble our case. Indeed, the Dirac spinor bundle S is a trivial one, constructed as a direct sum of two bundles of Chern numbers +1 and -1, $S = E(1) \oplus E(-1)$. If we tensor S by a line bundle of Chern number k, we have

$$S \otimes E(k) \sim E(k+1) \oplus E(k-1).$$

In our case, however, the generalized spinor bundle was different, as it was $E(N) \oplus E(-N)$, which is only stably equivalent to a trivial bundle, thus, the quasi-Dirac operator cannot be related to a twisted Dirac operator. The explicit identification of the basis of the linear space $\mathcal{V}_N$ (which we used to construct the spectral triple) with the sections of the line bundle of Chern index N (monopole harmonics) is discussed in [9, 14].

Of course, having the pseudodifferential operator D, one might look for its principal part, which should be of order one. This, however, will not necessarily be invariant under the rotation group and therefore breaking the rotational symmetry of the system. Hence the conclusion is that (apart from the case $N=1$) it is possible to construct and describe nontrivial topological sectors of fermions on the sphere (and therefore also on $\mathbb{R}^3\backslash\{0\}$) but only using pseudodifferential operators and a slightly extended notion of geometry. We leave further investigation of this operator for future work.

4. Conclusions

We have shown that within the framework of noncommutative geometry it is possible to construct nonequivalent spectral geometries on the two-dimensional sphere. The results open up several possibilities. First of all, it appears that the modification of the order-one condition has more profound consequences than it was believed. In this context, it would be nice to answer the question how the exactness of the order-one condition fixes the choice of the spinor bundle and the Dirac operator.

To study the analytic properties of the quasi-Dirac operators and to see whether there might by any differences from the classical case it shall be advisable to find the local (coordinate) expressions for the corresponding pseudodifferential operators.

Our result has also some implications for noncommutative geometries. We have shown that for the two spheres from each of the different K-homology classes it is possible to obtain unbounded Fredholm modules, with the order-one axiom satisfied up to compact operators. These are the naive $q = 1$ limits[4] of Dirac operators studied for quantum Podles spheres [12] and are in the noncommutative worlds, not easily distinguished from the deformation of the usual spectral geometry.

Acknowledgments

Part of the work was used for a EU project 'Researchers in Europe Initiative 2005', FP6-2004-Mobility-13, Proposal 019697 assigned to Guy Barrett and Andrew Elliott from St Robert of Newminister Catholic School and Sixth Form College, Durham, UK. The author would like to thank the Hausdorff Research Institute for Mathematics for hospitality and support.

References

[1] Atiyah M and Singer I M 1963 The index of elliptic operators on compact manifolds *Bull. Amer. Math. Soc.* **69** 422–33

[2] Connes A and Moscovici H 1995 The local index formula in noncommuative geometry *GAFA* **5** 174–243

[3] Connes A 1995 Noncommutative geometry and reality *J. Math. Phys.* **36** 619

[4] Connes A 2008 On the spectral characterization of manifolds arXiv:0810.2088

[5] Gracia-Bondía J M, Várilly J C and Figueroa H 2001 *Elements of Noncommutative Geometry* (Boston: Birkhäuser)

[6] Dąbrowski L, Landi G, Sitarz A, van Suijlekom W and Várilly J C 2005 The Dirac operator on $SU_q(2)$ *Commun. Math. Phys.* **259** 729–59

[7] Dabrowski L, Landi G, Paschke M and Sitarz A 2005 The spectral geometry of the equatorial Podles sphere *C. R., Math., Acad. Sci. Paris* **340** 819–22

[8] Dąbrowski L, Landi G, Sitarz A, Suijlekom W van and Várilly J C 2005 The local index formula for $SU_q(2)$ *K-Theory* **35** 375–94

[9] Mignaco J A, Sigaud C, Vanhecke F J and Silva A R Da 1997 The Connes–Lott program on the sphere *Rev. Math. Phys.* **9, 6** 689–717

[4] We consider here about a 'naive' $q = 1$ limit, as the problem of convergence of the spectral geometries is a hard question. A natural framework to study the notion of convergence could, for instance, be based on Gromov–Hausdorff–Rieffel distance constructed with the use of the Dirac operator. However, there are so far no results on whether this is possible for quantum spheres.

[10] Paschke M 2001 Über nichtkommutative geometrien, ihre Symmetrien und etwas Hochenergiephysik *Thesis* Mainz

[11] Sitarz A 2003 Equivariant spectral triples *Noncommutative Geometry and Quantum Groups (Banach Centre Publications)* vol 61, ed P M Hajac and W Pusz (Warszawa: IMPAN) pp 231–63

[12] Sitarz A 2008 Twisted Dirac operators over quantum spheres *J. Math. Phys.* **49** 033509

[13] Várilly Joseph 2006 An introduction to noncommutative geometry *EMS Series of Lectures in Mathematics (June)*

[14] Wu T T and Yang C N 1976 Dirac monopole without strings: monopole harmonics *Nucl. Phys.* B **107** 365–80